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Remarks on Differential Games of Survival*

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INTRODUCTION

In this paper we shall improve some results from [5] on differential games of survival. Mainly, we show, under weaker assumptions than in [5], that the upper value and the value exist, and that the Isaacs equation is satisfied almost everywhere. Our method is based upon considering a game of survival as a limiting case of games of fixed duration, and then applying recent results for games of fixed duration. This method was employed in [7] for deriving comparison theorems.

In Section 1 we prove the existence of upper and lower values, and (when the upper and lower Hamiltonians are equal) the existence of value. In Section 2 we prove that the upper value $V^+(t, x)$ satisfies the Isaacs equation almost everywhere.

Similar results (for a different notion of upper value) have been recently obtained by Elliott and Kalton [3, 4] by a different method, based upon extending the principle of dynamic programming to general stopping times. Our proofs are much simpler.

1. EXISTENCE OF UPPER VALUE AND OF VALUE

Consider a system of m differential equations

$$dx/dt = f(t, x, y, z), \tag{1.1}$$

an initial condition

$$x(t_0) = x_0\,, \tag{1.2}$$

and a payoff

$$P(y, z) = g(\bar{t}, x(\bar{t})) + \int_{t_0}^{\bar{t}} h(t, x, y, z)\, dt. \tag{1.3}$$

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Here Y and Z are compact subsets of some euclidean spaces R^p and R^q, respectively. The player y (z) chooses control functions $y(t)$ ($z(t)$), i.e., measurable functions with values in $Y(Z)$. The *capture time* $\bar{t}$, is the first value of t for which $(t, x(t))$ intersects a given closed ser F (in the (t, x)-space); F is called the *target set*, and it is assumed that $F \supset [T', \infty) \times R^m$ for some $T' > t_0$. The basic assumptions are:

(A_1) $f(t, x, y, z)$ is continuous in $[t_0, T] \times R^m \times Y \times Z$, for some $T > T'$.

(A_2) There exists a nonnegative function $k(t)$ with $\int_{t_0}^T k(t)\,dt < \infty$, such that

$$x \cdot f(t, x, y, z) \leqslant k(t)(|x|^2 + 1).$$

(A_3) For any $R > 0$ there exists a nonnegative function $k_R(t)$ with $\int_{t_0}^T k_R(t)\,dt < \infty$, such that

$$|f(t, x, y, z) - f(t, \bar{x}, y, z)| \leqslant k_R(t)\,|x - \bar{x}|$$

if $t_0 \leqslant t \leqslant T$, $y \in Y$, $z \in Z$, and $|x| < R$, $|\bar{x}| < R$.

(A_4) $g(t, x)$ and $h(t, x, y, z)$ are continuous in $[t_0, T] \times R^m \times Y \times Z$.

We shall need the following assumptions on F:

(F) F is a closed domain with C^2 boundary ∂F, and for all $(t, x) \in \partial F$

$$\nu_0 + \min_{z \in Z} \max_{y \in Y} \left\{ \sum_{j=1}^m \nu_j f_j(t, x, y, z) \right\} < 0$$

where $\nu = (\nu_0, \ldots, \nu_m)$ is the normal to ∂F at (t, x), pointing into the exterior of F.

($\tilde{F}$) The condition (F) holds and

$$\nu_0 + \min_{y \in Y} \max_{z \in Z} \left\{ \sum_{j=1}^m \nu_j f_j(t, x, y, z) \right\} < 0.$$

For $p \in R^m$, introduce the upper and lower Hamiltonian functions

$$H^+(t, x, p) = \min_{z \in Z} \max_{y \in Y} \{ f(t, x, y, z) \cdot p + h(t, x, y, z) \},$$
$$H^-(t, x, p) = \max_{y \in Y} \min_{z \in Z} \{ f(t, x, y, z) \cdot p + h(t, x, y, z) \}.$$

If $H^+ = H^-$ then we denote this function by H, and call it the Hamiltonian function.

THEOREM 1. *Let* (A_1)–(A_4) *and* ($\tilde{F}$) *hold, and let* $H^+ = H^-$. *Then the game associated with* (1.1)–(1.3) *has value.*

Proof. Consider first the case $g \equiv 0$. If $(t, x) \in F$, denote by $\rho(t, x)$ the distance from (t, x) to ∂F, and set

$$F_\mu = \{(t, x) \in F; \rho(t, x) < \mu\} \qquad (\mu > 0).$$

Let

$$\phi(r) = 1 - r \quad \text{if} \quad 0 \leqslant r \leqslant 1, \qquad \phi(r) = 0 \quad \text{if} \quad r > 1,$$

and define

$$h_\mu(t, x, y, z) = \begin{cases} h(t, x, y, z) & \text{if} \quad (t, x) \notin F, \\ h(t, x, y, z)\,\phi(\rho(t, x)/\mu) & \text{if} \quad (t, x) \in F. \end{cases}$$

Then h_μ is continuous and it vanishes for (t, x) in $F - F_\mu$.

Consider the payoff

$$P_\mu(y, z) = \int_{t_0}^{T} h_\mu(t, x, y, z)\, dt$$

and denote by $V^\delta(\mu)$, $V_\delta(\mu)$ the upper and lower δ-values of the game (of fixed duration) associated with (1.1), (1.2), and P_μ. Similarly define $V^+(\mu)$, $V^-(\mu)$. As in the proof of Lemma 1 in [7],

$$V_\delta \geqslant V_\delta(\mu) - \delta - C\mu \qquad (C \text{ constant}). \tag{1.4}$$

Similarly,

$$V^\delta \leqslant V^\delta(\mu) + \delta + C\mu.$$

Thus

$$0 \leqslant V^\delta - V_\delta \leqslant V^\delta(\mu) - V_\delta(\mu) + 2\delta + 2C\mu. \tag{1.5}$$

From Theorem 2.2.1 in [5],

$$| V^\delta(\mu) - V^+(\mu)| \leqslant \eta(\delta, \mu), \qquad | V_\delta(\mu) - V^-(\mu)| \leqslant \eta(\delta, \mu) \tag{1.6}$$

where $\eta(\delta, \mu) \to 0$ if $\delta \to 0$ (and μ is fixed). Hence,

$$0 \leqslant V^\delta - V_\delta \leqslant V^+(\mu) - V^-(\mu) + 2\delta + 2C\mu + 2\eta(\delta, \mu).$$

Observing that the upper and lower Hamiltonians corresponding to f, h_μ are equal, we conclude (by [1, 6]) that $V^+(\mu) = V^-(\mu)$.

Now, given any $\epsilon > 0$, choose $\mu = \epsilon/2C$ and $\delta^* = \delta^*(\epsilon)$ such that $\delta + \eta(\delta, \mu) < \epsilon$ if $\delta < \delta^*$. It follows that

$$0 \leqslant V^\delta - V_\delta < 3\epsilon \qquad \text{if} \quad \delta < \delta^*. \tag{1.7}$$

This implies (cf. [5, p. 90]) that the upper value $V^+ = \lim_{\delta \to 0} V^\delta$ and the lower value $V^- = \lim_{\delta \to 0} V_\delta$ exist and are equal.

So far we have assumed that $g \equiv 0$. Consider next the case where $g \not\equiv 0$, $g \in C^1$. Without loss of generality we may assume that $g(t_0, x_0) = 0$. Then

$$g(\bar{t}, x(\bar{t})) = \int_{t_0}^{\bar{t}} \frac{d}{dt} g(t, x(t))\, dt$$
$$= \int_{t_0}^{\bar{t}} \{g_t(t, x(t)) + \nabla_x g(t, x(t)) \cdot f(t, x(t), y(t), z(t))\}\, dt.$$

Consequently the payoff (1.3) is equal to the payoff

$$P(y, z) = \int_{t_0}^{\bar{t}} \tilde{h}(t, x, y, z)\, dt \tag{1.8}$$

where

$$\tilde{h}(t, x, y, z) = h(t, x, y, z) + g_t(t, x) + \nabla_x g(t, x) \cdot f(t, x, y, z). \tag{1.9}$$

Now the value exists by the special case $g \equiv 0$ considered above.

Suppose, finally, that $g(t, x)$ is any continuous function, and approximate it in compact subsets by a sequence of C^1 functions $g_j(t, x)$. Denote by V_j^δ, V_j^+ the upper δ-value and the upper value of the game associated with g_j. Similarly define $V_{\delta j}$, V_j^-. By a standard argument,

$$| V_j^\delta - V^\delta | \to 0 \qquad \text{if} \quad j \to \infty, \text{ uniformly with respect to } \delta.$$

Also, $| V_j^\delta - V_j^+ | \to 0$ if $\delta \to 0$ (for each j). Hence, for any $\epsilon > 0$ there is a j_0 such that

$$| V^\delta - V_{j_0}^+ | \leqslant | V^\delta - V_{j_0}^\delta | + | V_{j_0}^\delta - V_{j_0}^+ | < \epsilon + \eta_0(\delta)$$

where $\eta_0(\delta) \to 0$ if $\delta \to 0$. Similarly

$$| V_\delta - V_{j_0}^- | < \epsilon + \eta_0(\delta)$$

where j_0, $\eta_0(\delta)$ can be taken to be the same as before. Since $V_{j_0}^+ = V_{j_0}^-$, $V^\delta - V_\delta < 3\epsilon$ if δ is sufficiently small. This implies the existence of value.

For generalized pursuit–evasion games we have the following.

THEOREM 1′. *Let the conditions* (A_1)–(A_4) *and* (F) *hold, let* $H^+ = H^-$, *and let* $h \geqslant 0$, $g \equiv 0$. *Then the game associated with* (1.1)–(1.3) *has value.*

The proof is similar to the proof of Theorem 1.

Suppose now that (F) holds, then for any $(t, x) \in \partial F$ there is a value $z = z^*(t, x) \in Z$ such that

$$\nu_0 + \sum_{j=1}^{m} \nu_j f_j(t, x, y, z^*(t, x)) < 0 \qquad \text{for all} \quad y \in Y.$$

Similary, if $(\tilde{\mathrm{F}})$ holds, then for any $(t, x) \in \partial F$ there is a value $y = y^*(t, x) \in Y$ such that

$$\nu_0 + \sum_{j=1}^{m} \nu_j f_j(t, x, y^*(t, x), z) < 0 \qquad \text{for all} \quad z \in Z. \tag{1.10}$$

THEOREM 2. *Let* (A_1)–(A_4) *and* $(\tilde{\mathrm{F}})$ *hold. Then the game associated with* (1.1)–(1.3) *has upper value* V^+ *and lower value* V^-.

Proof. The proof of (1.4) is unchanged. Due to (1.10), we can now also establish the inequality

$$V_\delta \leqslant V_\delta(\mu) + \delta + C\mu.$$

Hence

$$| V_\delta - V_\delta(\mu)| \leqslant \delta + C\mu.$$

Using the second inequality in (1.6), we get

$$| V_\delta - V^-(\mu)| \leqslant \delta + C\mu + \eta(\delta, \mu). \tag{1.11}$$

Choosing $\mu = \epsilon/2C$ we conclude that for any $\epsilon > 0$ there is a $\delta^* = \delta^*(\epsilon)$ such that

$$| V_\delta - V_{\delta'} | < 3\epsilon \qquad \text{if} \quad \delta \leqslant \delta^*, \quad \delta' \leqslant \delta^*.$$

Hence $V^- = \lim_{\delta\to 0} V_\delta$ exists. The existence of V^+ is proved in the same way.

For generalized pursuit–evasion games we have the following.

THEOREM 2′. *Let* (A_1)–(A_4) *and* (F) *hold, and let* $h \geqslant 0$, $g \equiv 0$. *Then the game associated with* (1.1)–(1.3) *has upper value* V^+ *and lower value* V^-.

Remark 1. Theorems 1 and 1′ were proved in [5], by a different method, in the special case where

$$\begin{aligned} f(t, x, y, z) &= f^1(t, x, y) + f^2(t, x, z), \\ h(t, x, y, z) &= h^1(t, x, y) + h^2(t, x, z). \end{aligned} \tag{1.12}$$

Remark 2. Define a *strategy* α for y as a mapping from the set of all control functions of z to the set of all control functions of y, such that if, for some $s \in (t_0, T)$, $z(t) = \tilde{z}(t)$ for $t_0 \leqslant t \leqslant s$ then $(\alpha z)(t) = (\alpha \tilde{z})(t)$ for all $t_0 \leqslant t \leqslant s$. Similarly define a strategy β for z. This definition is generally different from the one adopted in [5]. Set

$$U^+ = \sup_\alpha \inf_z P(\alpha z, z),$$
$$U^- = \inf_\beta \sup_y P(y, \beta y),$$

where α and β vary over the sets of all strategies for the players y and z respectively, and $\inf_z$, $\sup_y$ are taken over all control functions $z(t)$, $y(t)$. As proved in [2], if (A_1)–(A_4) hold then (for games of fixed duration) $U^+ = V^+$ and $U^- = V^-$. We now note that an analog of Theorem 2 holds (with similar proof) with $V^\pm$ replaced by $U^\pm$. Here, instead of V_δ we work with

$$U_\delta = \inf_{\beta_1} \sup_{y_1} \cdots \inf_{\beta_n} \sup_{y_n} P(y_1, \beta_1 y_1, \ldots, y_n, \beta_n y_n)$$

where β_j is a strategy for z on the interval $(t_{j-1}, t_j]$. When $g \equiv 0$ we conclude (cf. (1.11)) that $U^- = \lim_{\mu \to 0} U^-(\mu)$. Since also $V^- = \lim_{\mu \to 0} V^-(\mu)$ and $U^-(\mu) = V^-(\mu)$, the relation $U^- = V^-$ follows. This relation can be extended to general $g(t, x)$ by the argument used in Theorem 1. We can thus state the following.

THEOREM 3. *Under the assumptions of either Theorem* 2 *or Theorem* 2′, $U^+ = V^+$, $U^- = V^-$. *Similarly, under the assumptions of Theorem* 1 *or* 1′, $U^+ = U^- = V^+ = V^-$.

2. The Isaacs Equation

Set $\Omega = [t_0, T] \times R^m - F$.

Denote by $V^+(t_0, x_0)$ the upper value of the game associated with (1.1)–(1.3). We shall now study the function $V^+(t, x)$. We need the assumption:

(A_5) $f(t, x, y, z)$, $h(t, x, y, z)$ and $g(t, x)$ are uniformly Lipschitz continuous in (t, x) in compact subsets.

THEOREM 4. *Let* (A_1)–(A_5) *and* (F) *hold. Then* $V^+(t, x)$ *is uniformly Lipschitz continuous in compact subsets of* Ω, *and it satisfies almost everywhere in* Ω *the Isaacs equation*

$$(\partial V^+/\partial t) + H^+(t, x, \nabla_x V^+) = 0. \tag{2.1}$$

Finally, V^+ *is continuous in* $\bar{\Omega}$ *and* $V^+ = g$ *on* ∂F.

Proof. For any compact subset $K \subset \Omega$ there is a constant C such that

$$| V^\delta(t, x) - V^\delta(\bar{t}, \bar{x})| \leqslant C(| t - \bar{t} | + | x - \bar{x} |) \tag{2.2}$$

for all $(t, x) \in K$, $(\bar{t}, \bar{x}) \in K$ and for all δ. This is precisely the assertion (5.4.12) in [5], and the proof in [5] requires only the conditions (A_1)–(A_5) and $(\tilde{F})$ of the present paper. [The condition (1.12) is not required in this proof.] Taking $\delta \to 0$ in (2.2), we conclude that $V^+(t, x)$ is uniformly Lipschitz continuous in compact subsets of Ω. By a theorem of Rademacher [5, p. 122], V^+ is differentiable almost everywhere.

Set

$$W^n(t, x) = V^\delta(t, x) \qquad (\delta = (T - t)/n).$$

Then from (2.2) and the proof of Theorem 3.6.3 in [5, p. 99] we deduce that, as $n \to \infty$,

$$W^n(t, x) \to V^+(t, x) \tag{2.3}$$

uniformly with respect to (t, x) in compact subsets of Ω.

Using the differentiability of V^+ and (2.3), we can now proceed to establish (2.1) by the same proof as that of Theorem 4.2.1 (cf. also Theorems 4.2.7 and 5.5.1) of [5]. Finally, the assertion

$$V^+(t, x) \to g(s, y) \qquad \text{if} \quad (t, x) \to (s, y) \in \partial F$$

is easily verified, using the condition $(\tilde{F})$ (cf. [5, p. 204, Problem 3]).

For generalized pursuit–evasion games, Theorem 4 remains true also when (F) is replaced by (F).

3. Additional Results

Remark 1. In [7] we proved a comparison theorem for games of survival with $g \equiv 0$. The case of general g follows by first considering the case $g \in C^1$, replacing (1.3) by (1.8). The case of continuous g then follows by approximation. In [7] we assume that (1.12) holds. However, in general, when $H^+ \neq H^-$, the same proof gives a comparison theorem for V^+, V^-. Thus, if

$$U_t + H^+(t, x, \nabla_x U) \leqslant 0 \text{ in } \Omega,$$
$$U \geqslant g \text{ on } \partial F,$$

then $V^+(t_0, x_0) \leqslant U(t_0, x_0)$. Here it is assumed that (A_1)–(A_4) hold, that $(\tilde{F})$ holds, that U is in C^1, and that $\nabla_x U$ is piecewise continuously differentiable. Similar comparison theorems for U^+ have recently been proved by Elliott and Kalton [3] by a different method. They do not assume the condition ($\tilde{F}$) and the existence of first derivatives of $\nabla_x U$. Another proof, similar to theirs (for comparing $\overline{\lim}_{\delta \to 0} V^\delta$ with U), can also be constructed by suitably modifying the proof of Theorem 4.3.4 in [5].

Remark 2. Consider a differential game in restricted phase set X. Suppose all the assumptions made in the various existence theorems of [5, Chap. 6] are satisfied, except for (1.12) which is dropped out. Assume also that the first relation in (1.12) holds near ∂X. Then we can still prove [by the methods of that chapter] that V^+ and V^- exist and that they satisfy the Isaacs equation. If $H^+ = H^-$ then we can prove that $V^+ = V^-$, by approximating the game by a sequence of games of fixed duration (as in [7]) for which the value exists.

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